of Ref. 2 and those of the previous procedure for axisymmetric loads.

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Terminal Guidance for Continuous Powered Space Vehicles

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In the investigation presented here, a method for determining the guidance procedure for a continuous powered space vehicle very similar to that suggested in Ref. 1 is proposed. One advantage of the proposed method is that the arbitrary weighting matrix, which is a characteristic of the scheme discussed in Ref. 1, is replaced by a uniquely specified matrix. Furthermore, a state prediction scheme is given which is general enough to handle all terminal constraint requirements. The control procedure is developed by requiring that the change in the value of the original performance index be a minimum. That is, the control deviation program seeks to reoptimize the performance index associated with the nominal trajectory.

If $x(t)$ represents the deviation in the state and $u(t)$ represents the deviation in the control, then the state and control program associated with the actual trajectory can be specified as $X(t) = X^*(t) + x(t)$ and $U(t) = U^*(t) + u(t)$ where $X^*(t)$ and $U^*(t)$ represent a reference state history and control program. By expanding the nonlinear equations of motion in a Taylor's series about the reference trajectory at each point in time, the following expression for the state deviation histories can be obtained:

$$\dot{x} = Ax + Bu \quad (1)$$

where $A = (F_X)^*$ and $B = (F_U)^*$[2]. The symbol ( )* indicates that the quantity in the parenthesis is evaluated on the reference trajectory.

Assume that the guidance maneuver is to be carried out in such a way that the change in the original (or nominal) performance index is minimized. If the reference trajectory is optimal, the increase in the value of the performance index associated with following some trajectory other than the optimal reference trajectory can be approximated by

$$\Delta P = \int_{t_0}^{t_f} E(X^*, \dot{X}^*, \dot{X}, t)dt$$

where E is the Weierstrass E-function (Ref. 3). For a Mayer formulation of the optimization problem, $E = H(X^*, U, t) - H(X^*, U^*, t)$. A necessary condition for a minimum value of the nominal performance index is $E \geq 0$. For a sufficiently small deviation in the nominal control program, E can be approximated by the following expression $E = (U - U^*)H^*_{UU}\cdot(U - U^*) = \frac{1}{2}u^T H_{UU}{}^* u$ where $H_{UU}{}^*$ is an $m \times m$ matrix. With this approximation, the expression for ΔP becomes

$$\Delta P = \int_{t_0}^{t_f} \tfrac{1}{2}(u^T H_{UU}{}^* u)dt \quad (2)$$

The problem to be considered can be stated now as follows: Determine the control deviation program $u(t)$ which will minimize ΔP subject to the differential equations $\dot{x} = Ax + Bu$ while satisfying the terminal conditions $M(x_f, t_f) = 0$. The solution to this problem requires that the following conditions be satisfied:

$$\dot{x} = Ax + Bu \qquad (x)_0 = x_0 \qquad \dot{\lambda}^T = -\lambda^T A$$
$$(\lambda^T - \nu^T M_x)_{t_f} dx_f = 0 \qquad 0 = H_{UU}{}^* u + B^T\lambda$$
$$(\nu^T M_t + \lambda^T \dot{x})_{t_f} dt_f = 0 \quad (3)$$

From the third of Eqs. (3), the optimum control deviation program is given as $u(t) = -(H_{UU}{}^*)^{-1}B^T\lambda(t)$. A complete solution requires only that the $\lambda(t)$ be determined. The $\lambda(t)$ are governed by the second of Eqs. (3). They can be expressed as $\lambda(t) = \Phi^T(t, t_f)\lambda(t_f)$ where $\Phi^T(t, t_f)$ is determined by integrating the second of Eqs. (3), backward from t_f, n times with the starting conditions $\lambda_{ij} = \delta_{ij}$, $(i, j, = 1 \ldots n)$ (δ_{ij} is the Kronecker δ). With these conditions, $\Phi(t_f, t_f) = I$, the identity matrix. If dx_f is not zero, then from the fourth of Eqs. (3), $\lambda^T(t_f) = \nu^T M_{x_f}$. Hence, $u(t)$ can be expressed as follows

$$u(t) = -(H_{UU}{}^*)^{-1}B^T\Phi(t, t_f)M_{x_f}{}^T\nu \quad (4)$$

Note that the constants ν are unspecified.

If the first of Eqs. (3) is premultiplied by λ^T and added to the second of Eqs. (3) postmultiplied by x, the resulting expression will yield

$$(\lambda^T x)_{t_f} = (\lambda^T x)_{t_0} + \int_{t_0}^{t_f} \lambda^T B\, u\, dt \quad (5)$$

The expressions for $\lambda^T(t)$ and $u(t)$ can be substituted into Eq. (5) to obtain the following expression:

$$\lambda_f{}^T x_f = \lambda_f{}^T[\Phi(t_0, t_f)x_0 - J(t_0, t_f)M_{x_f}{}^T \nu] \quad (6)$$

where

$$J(t, t_f) = \int_t^{t_f} [\Phi(\tau, t_f)B\, H_{UU}{}^{*-1}\, B^T\Phi^T(\tau, t_f)]d\tau$$

Equation (6) can be used to determine the constants ν and also as the starting point for developing the state prediction scheme.

Consider the fixed final time problem where t_f is specified to be the final time associated with the nominal trajectory. If the conditions

$$(\lambda_j{}^T)_{t_f} = (\partial M^j/\partial x_f) \qquad j = 1, \ldots p \quad (7)$$

are used in Eq. (6) the resulting set of p independent conditions can be expressed

$$M_{x_f} x_f = M_{x_f}[\Phi(t_0, t_f)x_0 - J(t_0, t_f)M_{x_f}{}^T\nu] \quad (8)$$

where $M_{x_f} = (\partial M/\partial x_f)$. Now, provided that $[M_{x_f}J(t_0, t_f)M_{x_f}{}^T]$ is nonsingular, Eq. (8) can be solved for the p unknown constants ν, i.e.,

$$\nu = -[M_{x_f}J(t_0, t_f)M_{x_f}{}^T]^{-1}M_{x_f}[x_f - \Phi(t_0, t_f)x_0] \quad (9)$$

The following form for the optimal control deviation program

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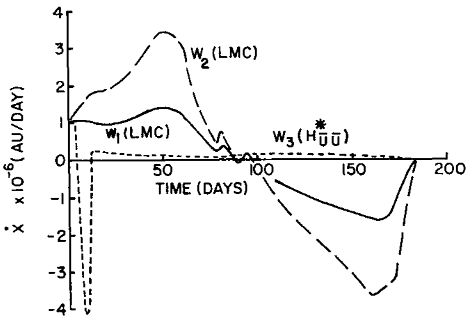

Fig. 1 State deviations for $\dot{x}(0) = 10^{-6}$ a.u./day.

can be obtained by substituting Eq. (9) into Eq. (4).

$$u(t) = H_{UU}^{*-1}B^T\Phi(t, t_f)M_{x_f}{}^T \times [M_{x_f}J(t_0, t_f)M_{x_f}{}^T]^{-1}M_{x_f}[x_f - \Phi(t_0, t_f)x_0] \quad (10)$$

For the special case where all of the terminal conditions are specified, i.e., if $p = n$, then $[M_{x_f}J(t_0, t_f)M_{x_f}{}^T]^{-1} = M_{x_f}{}^{T-1} J(t_0, t_f)^{-1}M_{x_f}{}^{-1}$ and Eq. (10) will reduce to the following expression

$$u(t) = H_{UU}^{*-1}B^T\,\Phi(t, t_f)J^{-1}(t_0, t_f) \times [x_f - \Phi(t_0, t_f)x_0] \quad (11)$$

The variable final time problem is discussed in Ref. (4). Consider now the problem of determining a means for predicting the deviation in the state of a vehicle subsequent to the initiation of a control deviation program $u(t)$ as described by Eq. (10). The predicted value of the state deviation at some time $t_1 > t_0$ must be compared with the value of the state deviation estimated by the navigation procedure to determine whether or not the prescribed control deviation program has been implemented correctly.

The state prediction scheme can be obtained by substituting the conditions $(\lambda_i{}^T)_{tf} = (\partial x^i/\partial x)_{tf}$, $i = 1, \ldots, n$, into Eq. (6) to obtain the following n independent solutions [i.e., one for each $(\lambda_i{}^T)_{tf}$]

$$x_f = \Phi(t_0, t_f)x_0 - J(t_0, t_f)M_{x_f}{}^T\nu \quad (12)$$

For the fixed final time problem, the ν is given by Eq. (9). Now, let $x(t_1)$ be the value of the state deviation at some time $t_1 > t_0$ which results from the control deviation program $u(t)$ initiated at (x_0, t_0). For a unique control deviation program $u(t)$, x_f must be unique. Hence it follows that

$$x_f = \Phi(t_1, t_f)x(t_1) - J(t_1, t_f)M_{x_f}{}^T\nu \quad (13)$$

Now eliminating x_f from Eqs. (12) and (13) and solving for $x(t_1)$ leads to the following expression

$$x(t_1) = \Phi(t_1, t_f)^{-1}\{\Phi(t_0, t_f)x_0 - [J(t_0, t_f) - J(t_1, t_f)]M_{x_f}{}^T\nu\} \quad (14)$$

Equation (14) represents a general state prediction scheme that can be used with the general set of terminal conditions, $M(x_f, t_f) = 0$. If the terminal value of each of the nominal state variables is specified and if it is required that the nominal state be reattained by the control scheme, i.e., if $x_f = 0$, then Eq. (14) will reduce to

$$x(t) = \Phi^{-1}(t_f, t)J(t_f, t)J^{-1}(t_f, t_0)\Phi(t_f, t_0)x_0 \quad (15)$$

for $t_0 \leq t \leq t_f$. This form for the prediction scheme has been obtained by Friedlander in a modification of the prediction scheme given in Ref. 5.

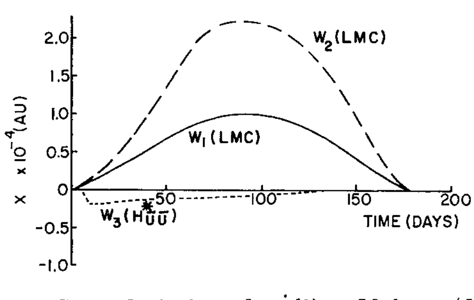

Fig. 2 State deviations for $\dot{x}(0) = 10^{-6}$ a.u./day.

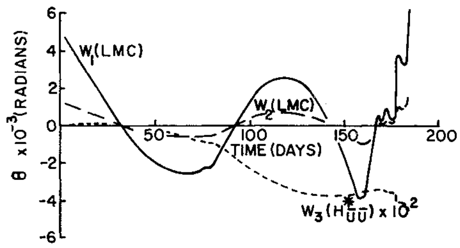

Fig. 3 Control deviations for $\dot{x}(0) = 10^{-6}$ a.u./day.

The theory discussed previously has been applied to the problem of controlling the state deviations for a three-dimensional low-thrust space vehicle flying an optimal 184-day transfer trajectory to Mars where the control program is chosen to minimize the propellant consumed. Only the heliocentric portion of the transfer is considered, and, the vehicle is assumed to move in an inverse square gravitational field influenced only by the attraction of the sun.

The details of the numerical study are given in Refs. 2 and 4. Figure 1 shows the state deviation in the x component of velocity following an error of 10^{-6} a.u./day (a.u. = astronomical units) in the x component of velocity for three control-program weighting matrices [i.e., see Eq. (4)].

$$W_1(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad W_2(t) = \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$W_3(t) = \begin{bmatrix} (\partial^2H/\partial\theta^2)^* & (\partial^2H/\partial\theta\partial\psi)^* \\ (\partial^2H/\partial\theta\partial\psi)^* & (\partial^2H/\partial\psi^2)^* \end{bmatrix}$$

where $\psi(t)$ and $\theta(t)$ are the in-plane and out-of-plane thrust orientation angles, respectively. Figure 2 shows the variations with time of the x component of displacement for the same initial error. Similar results are given in Ref. 2 and Ref. 4 for the y and z directions. Figure 3 shows the control deviation programs that caused the results in Figs. 1 and 2. It is interesting to note that during the thrust turn-around period for the nominal trajectory (80 to 100 days) the controlled state deviations for the $W_1(t)$ and $W_2(t)$ weighting matrices have rapid and undesirable oscillations. This disturbance is smoothed out when H_{UU}^* is used as the weighting matrix in the control scheme.

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